Problem Set 8

Discrete Mathematics

Due on the 5th of April, 2024

(10 pts)   1. Including the instructor, there are 32 people in our class. Prove that two of these people were born on the same day of the month.

(10 pts)   2. As of the 28th of March, 2024, there are over 8.1 billion people living on Earth.[1] A person's heart will beat no more than 7×10^9 times over their lifespan. Show that there are two currently-living people on Earth whose hearts have beat the exact same number of times.

[1] For simplicity, you may assume the human population will only monotonically increase over time from the point this problem set was assigned.

(20 pts)   3. Let $n \in \mathbb{N}_+$ and consider $\mathcal{A} \subseteq \mathbb{N}$ such that $|\mathcal{A}| = n + 1$. Prove there exist $x, y \in \mathcal{A}$ with $x \neq y$ such that $n \mid x - y$.

(20 pts)   4. Consider $\mathcal{S} := \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$ and $\mathcal{X} \subseteq \mathcal{S}$ with $|\mathcal{X}| \geqslant 9$. Show that there exist three *distinct* elements $x_1, x_2, x_3 \in \mathcal{X}$ such that $x_1 + x_2 + x_3 = 40$.

(20 pts)   5. Recall that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n, k \in \mathbb{N}$ when $k \leqslant n$.

   (a) Show $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ where $k \leqslant n$.

   (b) Show $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ for all $n, k \in \mathbb{N}$ where $k \leqslant n$.

(20 pts)   6. Prove that $|\mathbb{P}(X)| = 2^{|X|}$ for any finite set X.