Problem Set 3

Discrete Mathematics

Due on the 12th of February, 2024

In addition to the axioms and rules of inference, you may rely on: all proven theorems, *Implication Elimination*, *Hilbert's First & Second Axioms*.

(10 pts) 1. Prove each of the following statements for any propositions φ, ψ, ξ.

(a) $(\varphi \to \psi), (\psi \to \xi) \vdash (\varphi \to \xi)$ — *Hypothetical Syllogism*

(b) $\varphi, \psi \vdash \varphi \wedge \psi$ — *Conjunction Introduction*

(40 pts) 2. Prove each of the following statements for any propositions φ, ψ, ξ.

(a) $\vdash \varphi \to \varphi$

(b) $\vdash (\neg\varphi \to \varphi) \to \varphi$ — *Consequentia Mirabilis*, a.k.a. *Lex Clavia*

(c) $\vdash \neg\varphi \to (\varphi \to \neg\psi)$ — *Ex Contradictione Quodlibet*

(d) $\varphi \wedge \psi \vdash \varphi$ — *Conjunction Elimination*

(e) $\vdash \top$ — *The Truth Theorem*

(30 pts) 3. Prove each of the following statements for any propositions φ, ψ, ξ, χ.

(a) $\varphi \vdash (\varphi \vee \psi)$ — *Disjunction Introduction*, a.k.a. *Addition*

(b) $(\varphi \to \xi), (\psi \to \xi), (\varphi \vee \psi) \vdash \xi$ — *Disjunction Elimination*, a.k.a. *Proof by Cases*

(c) $\varphi, \neg\varphi \vdash \psi$ — *Ex Falso Quodlibet*, a.k.a. *Explosion*

(d) $(\varphi \vee \psi), \neg\varphi \vdash \psi$ — *Disjunctive Syllogism*

(e) $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash \xi \vee \chi$ — *Constructive Dilemma*

(10 pts) 4. Let $\mathcal{L}$ be a binary predicate. Prove the following statement.[1]

$$\vdash \neg\exists x \forall y (\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$$

[1] *Hint: try a proof by contradiction.*

(10 pts) 5. Consider a universe of discourse consisting of every natural number. Recall that a positive integer is *prime* when it has *exactly two* positive divisors: one and itself.

As a fun side note: 2 is a prime number.

Let $\omega(x) :=$ *"x is an odd number."*
Let $\pi(x) :=$ *"x is a prime number."*

Further, suppose the following statements only contain propositions.

(a) Prove φ, where φ is the statement $\varphi \vdash \forall x\big(\omega(x) \to \pi(x)\big)$.

(b) Prove $\forall x\big(\omega(x) \to \pi(x)\big)$.