Exam 1.p.3

Discrete Mathematics

8^{th} of March, 2024, Spring

1 *Answer the following questions by marking either True or False.*

Make sure your name and NetID is written at the top of each page of this exam. Write legibly. Read carefully.

1. $\{\varnothing\}$ is a subset of every set.
   - ○ *True*
   - ○ *False*
2. $\bigcup \mathfrak{s}(n) = n$ for every natural number $n \in \mathbb{N}$.
   - ○ *True*
   - ○ *False*
3. $\bigcup \mathfrak{s}(x) = x$ for every set x.
   - ○ *True*
   - ○ *False*
4. $(\forall \mathcal{A} \subseteq \mathbb{Z})\big(\mathcal{A} \neq \varnothing \Rightarrow (\exists a \in \mathcal{A})(\forall b \in \mathcal{A})(a \leqslant b)\big)$.
   - ○ *True*
   - ○ *False*
5. This sentence has no truth value.
   - ○ *True*
   - ○ *False*
6. The statement φ is a proposition, where $\varphi :=$ "$\varphi \to (\varnothing \in \varnothing)$."
   - ○ *True*
   - ○ *False*
7. $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})\Big(\big((x \leqslant y) \wedge (x \neq y)\big) \Rightarrow x \in y\Big)$.
   - ○ *True*
   - ○ *False*
8. Every natural number is the successor of another natural number.
   - ○ *True*
   - ○ *False*
9. $\gcd(a,b) \leqslant \operatorname{lcm}(a,b)$ for every $a, b \in \mathbb{Z}$.
   - ○ *True*
   - ○ *False*
10. If $\varphi(\cdot)$ is a predicate with one free variable, then $\{z \mid \varphi(z)\}$ is a set.
    - ○ *True*
    - ○ *False*

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2 *Any use of logical axioms, rules of inference, or theorems of logic must be explicitly stated. You may not appeal to truth tables.*

Show that $(\alpha \vee \beta), (\beta \rightarrow \alpha), (\alpha \rightarrow \gamma) \vdash \gamma$ for any propositions α, β, γ.

3 *You may rely on any theorems we have proven.*

Show that $\forall x \forall y (x \subseteq y \Rightarrow x \subseteq x \cap y)$.

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4 *You may only rely on the assumptions in the margin.*

1. $0 \in \mathbb{N}$.
2. $(\forall n \in \mathbb{N})\big(\mathfrak{s}(n) \neq 0\big)$.
3. $(\forall n \in \mathbb{N})\big(\mathfrak{s}(n) \in \mathbb{N}\big)$.
4. $(\forall n, m \in \mathbb{N})\big(\mathfrak{s}(n) = \mathfrak{s}(m) \Rightarrow n = m\big)$.
5. The mathematical induction schema.
6. $(\forall a, b, c \in \mathbb{N})(a \cdot (b + c) = (a \cdot b) + (a \cdot c))$.

1. Provide a *recursive* definition of addition for natural numbers.

2. Provide a *recursive* definition of multiplication for natural numbers.

3. Show that $(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(\forall c \in \mathbb{N})\big(a \cdot (b \cdot c) = (a \cdot b) \cdot c\big)$.

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5 *You may rely on any proven theorems.*

1. Provide a *recursive* definition of the Fibonacci sequence $\mathcal{F} : \mathbb{N} \to \mathbb{N}$.

2. Show $\sum\limits_{i=0}^{n} \mathcal{F}(2i+1) = \mathcal{F}(2n+2)$ for all $n \in \mathbb{N}$.